The Schwartz-Christoffel Formula

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Where We Left Off

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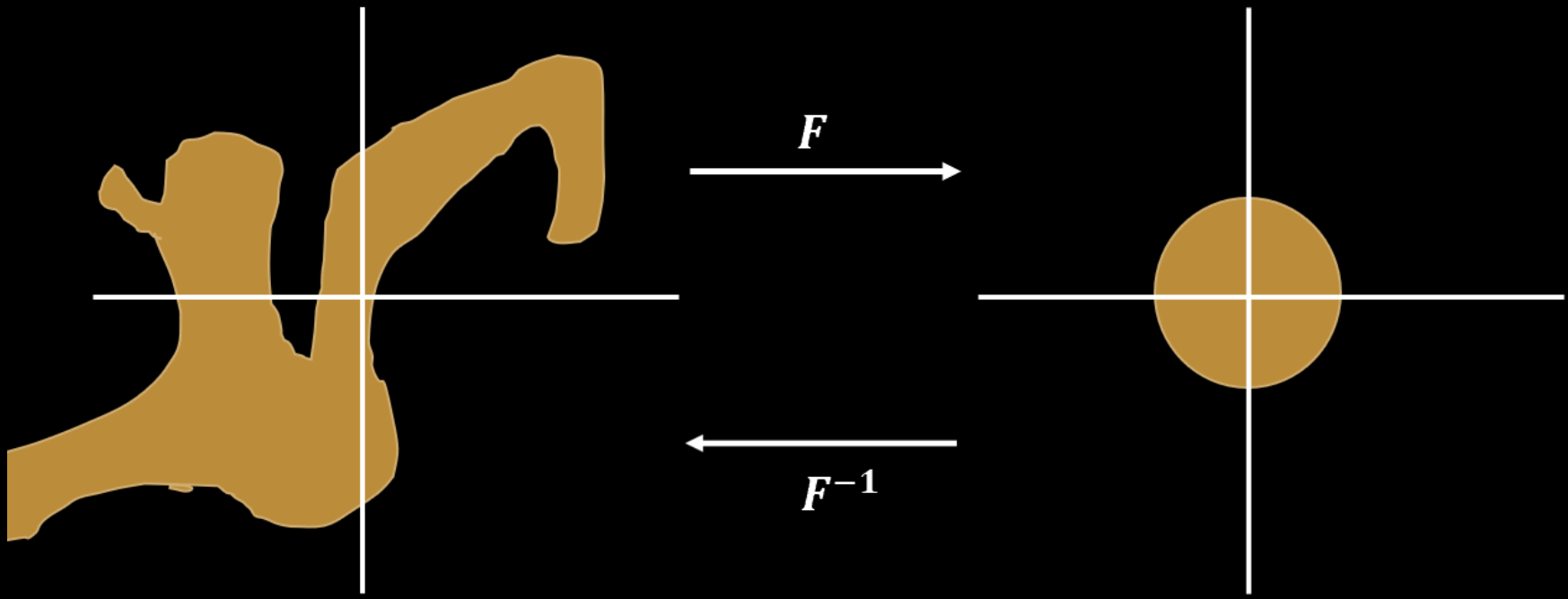
Wah, wah, wah. . . ..

A Brief Illustration of Our Problem



https://memegenerator.net/instance/71691819/sad-kitty25-the-look-you-give-someone-when-you-want-something

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More Background

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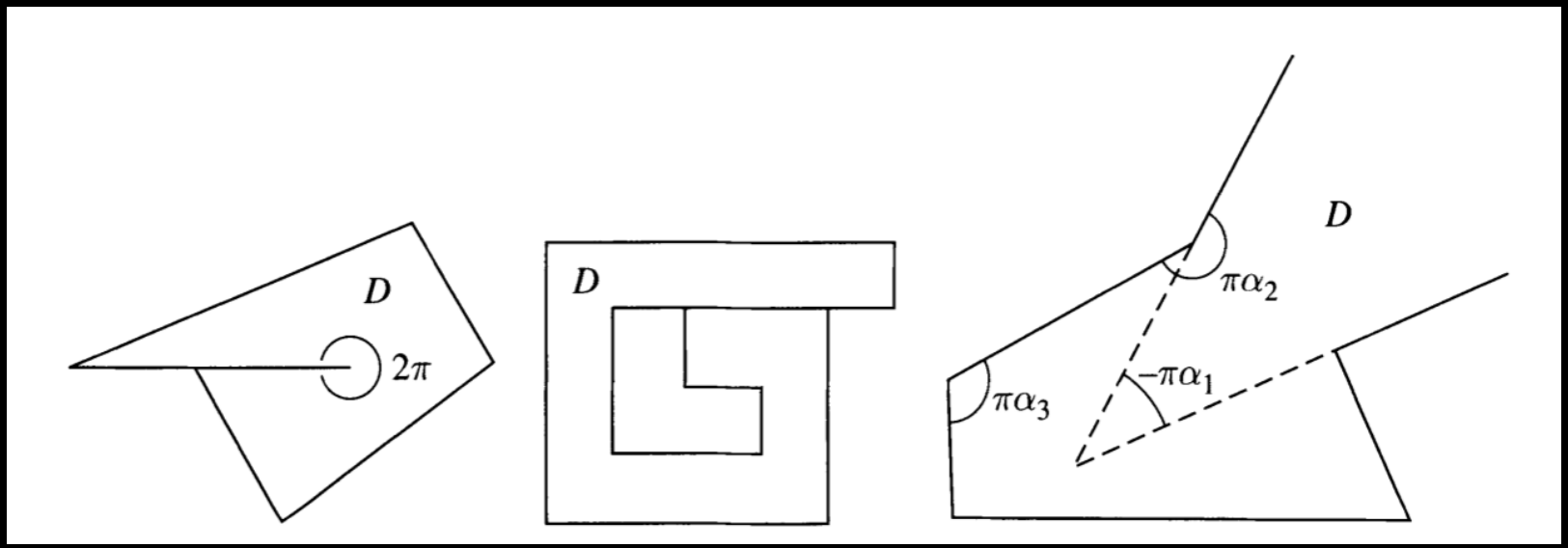
Examples:



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Let g be a conformal mapping from the upper half plane Π^+ onto a polygonal domain of n sides which has interior angles with measures $\alpha_1\pi, ..., \alpha_n\pi$ and let $a_1, a_2, ..., a_n$ be the points on the real axis mapped onto the vertices of the polygon.

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$$g(w) = A \int_{w_0}^{w} (z - a_1)^{\alpha_1 - 1} ... (z - a_n)^{\alpha_n - 1} dz + B$$

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(taken from Bruna, Cufi, p. 361)

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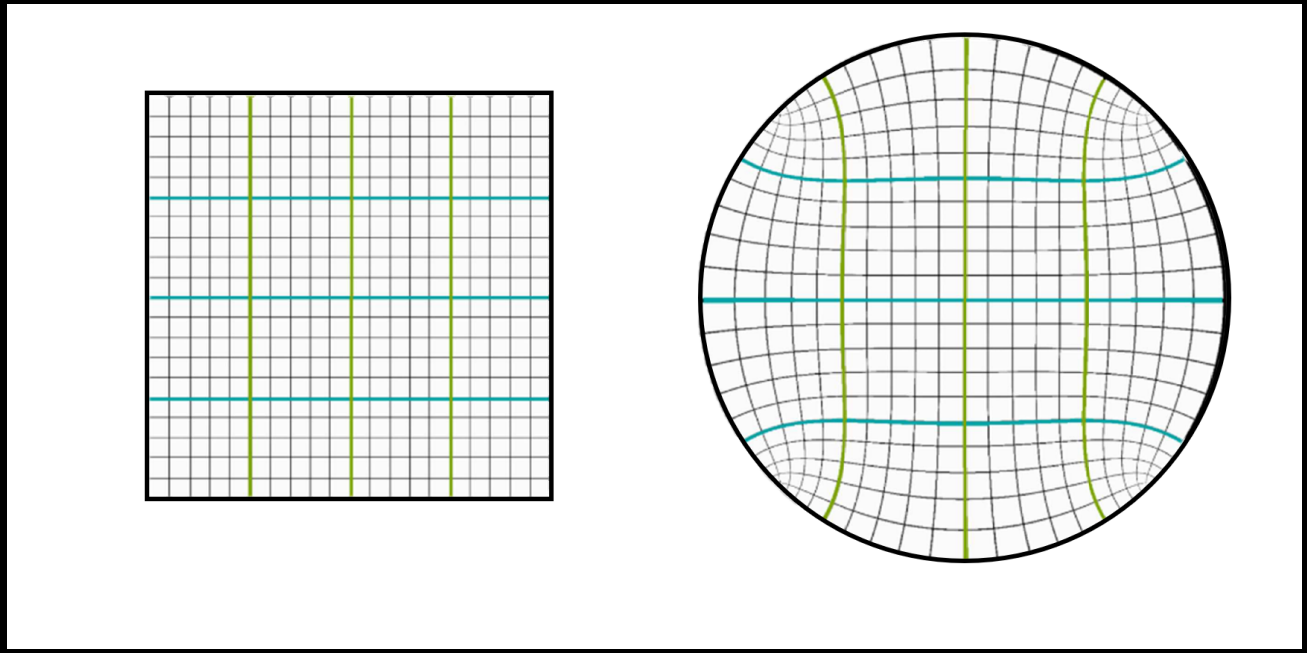
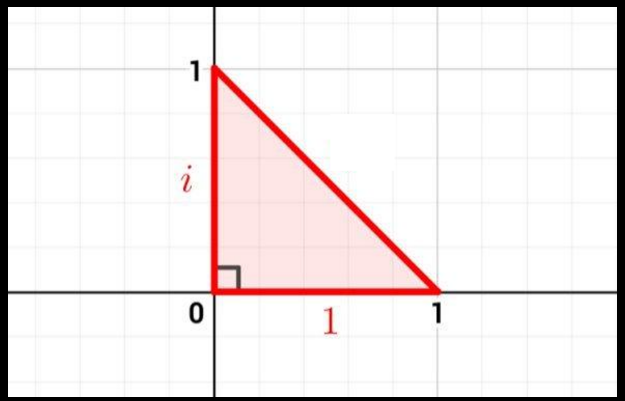
Conformal



Taken from Fong's The Conformal Hyperbolic Square and Its Ilk

Example with Triangle

Consider the triangle with vertices $0, 1, i$ with interior angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$.



From lecture, we have a biholomorphism from the half plane to the unit disk. So, finding a biholomorphism from the half plane to the triangle is sufficient.

Example with Triangle

Let $g(z)$ map the upper half plane to our polygon with $g(0) = 0$, $g(1) = 1$, and $g(a) = i$ for some $a > 1$. Then, by the Schwartz-Christoffel Formula

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A can be found by using the fact that $g(1) = 1$.

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$$= \frac{A_z}{\sqrt{z^2-1}}$$

Computer Illustration

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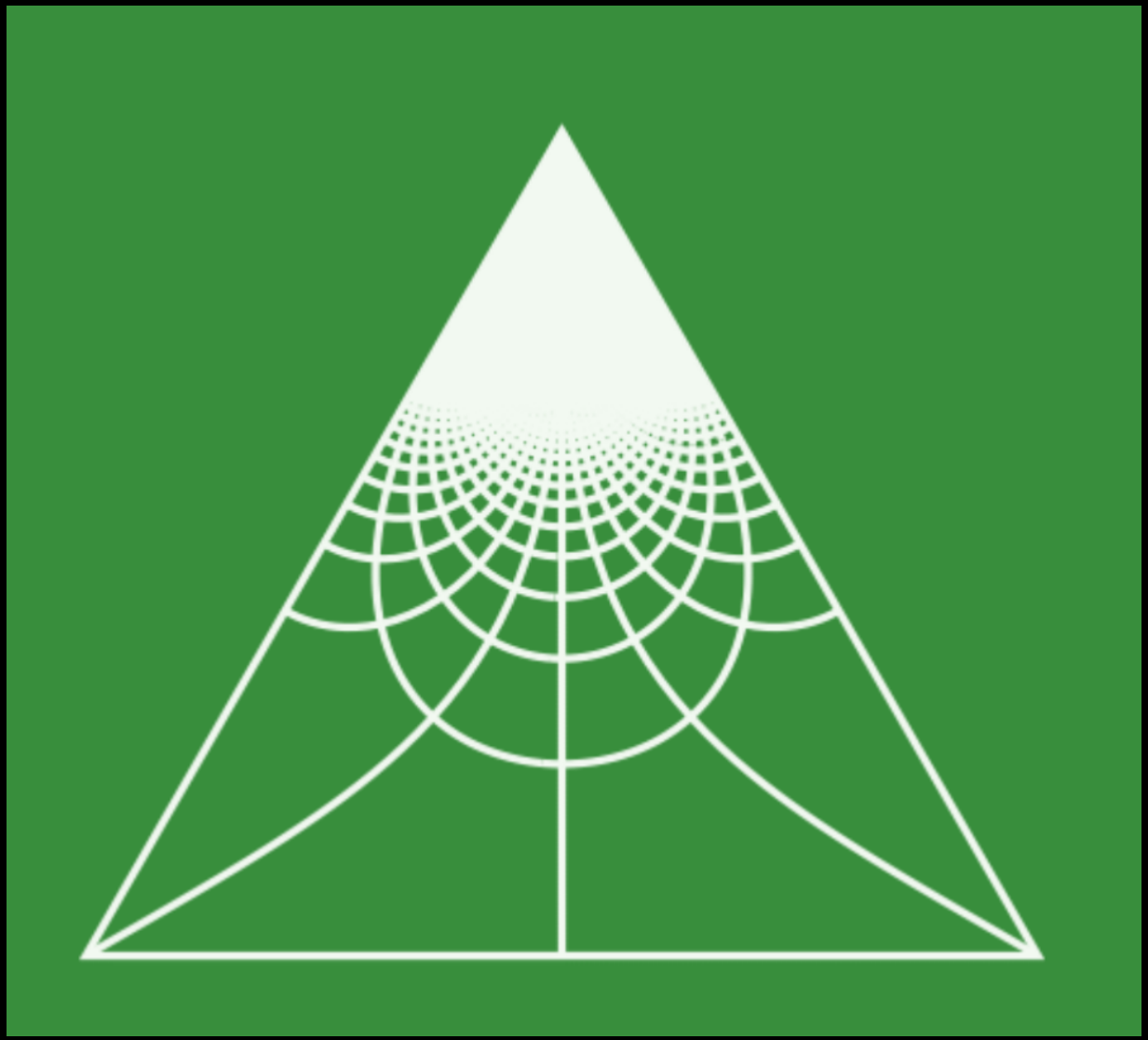
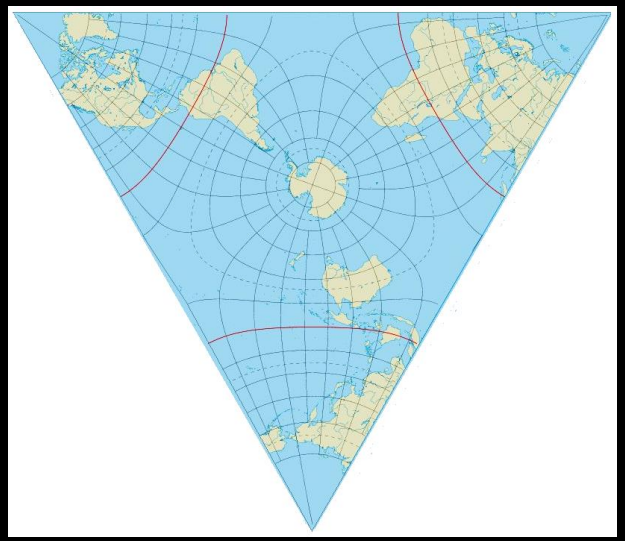
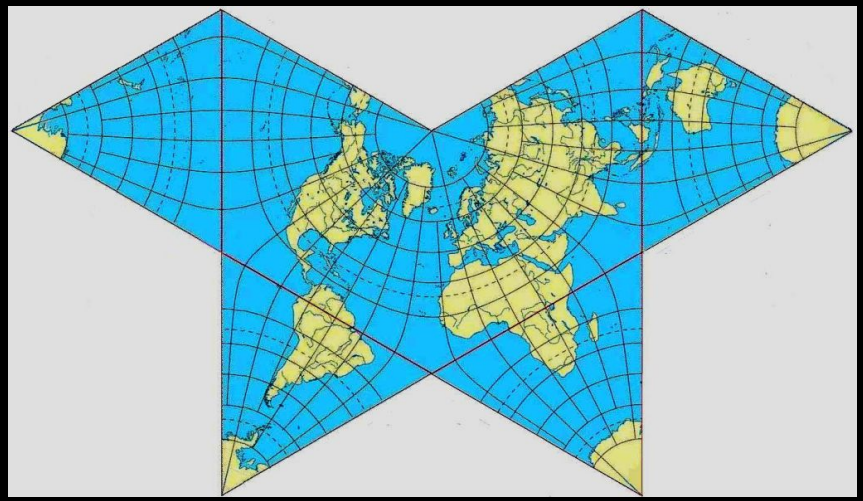


A Cool Application





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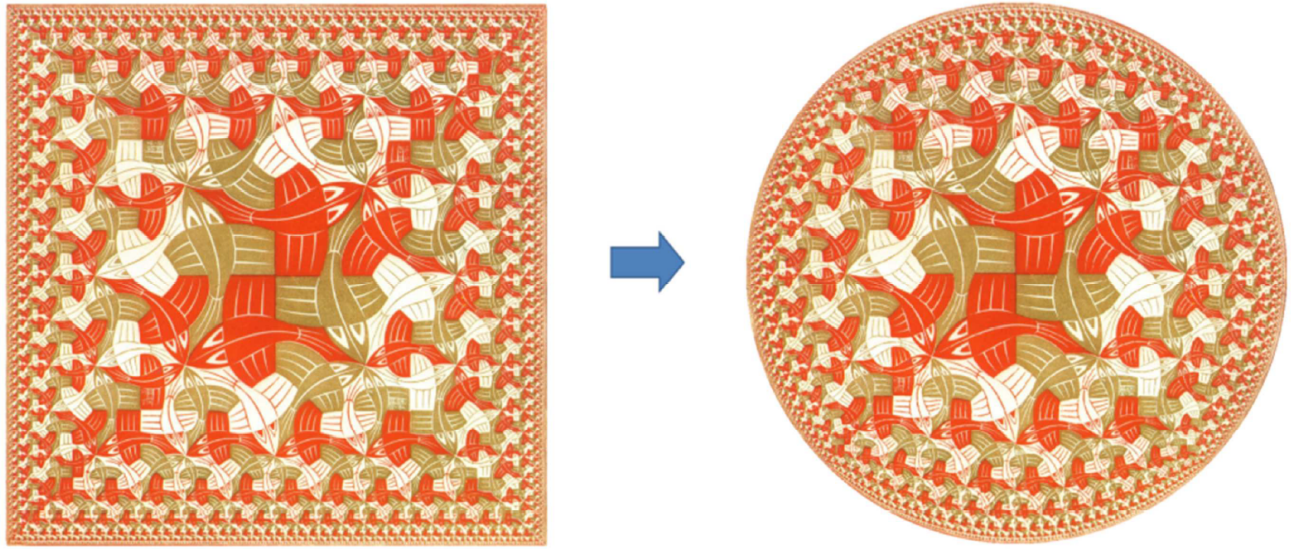

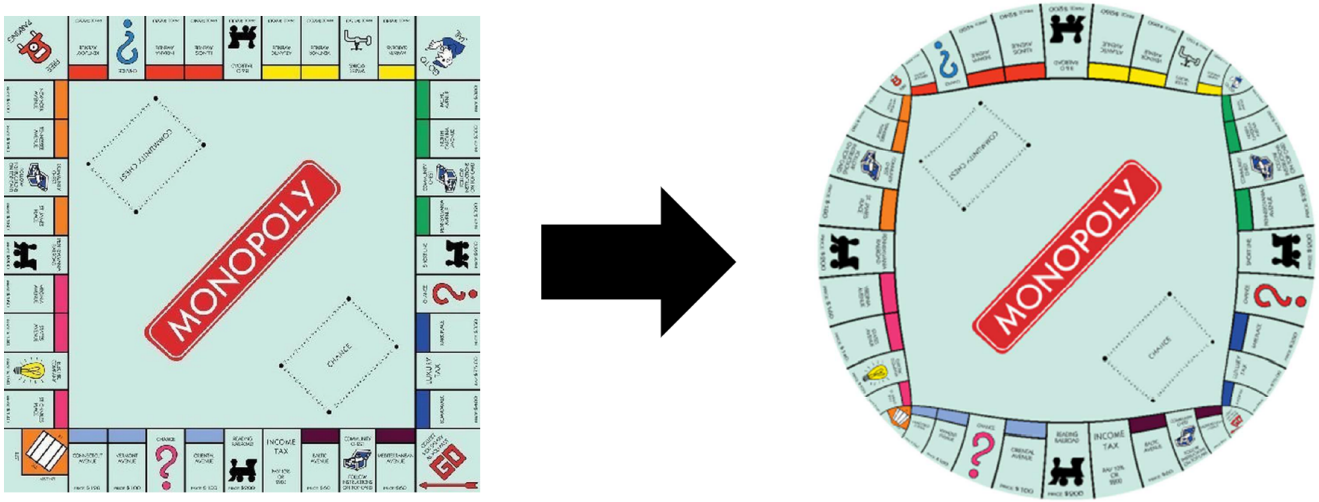

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