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Russell's Paradox and Domains

Russell's Paradox considers R, the set of all sets that do not contain themselves. A contradiction arises when we ask whether R contains itself. If $R \in R$ then R contains itself meaning that $R \notin R$, by its definition. On the other hand, if $R \notin R$ then R does not contain itself meaning that it must be in R, by the definition of R. Thus, $R \in R$. Either way, we get a contradiction. Modern mathematics uses the Zermelo-Fraenkel axioms with the axiom of choice (ZFC) to avoid this paradox. However, this paradox deserves attention because it may be solved and free mathematicians, logicians, and philosophers from the restrictions they have imposed on themselves to avoid Russell's Paradox.

Russell's Paradox is resolved by arguing that the function $F(x)$, which returns true if x contains itself and is false otherwise, does not have R in its domain. Thus, the set R is invalid. This paper discusses how this solution solves many other paradoxes of self reference including Grelling's Paradox and the Liar Paradox while still allowing non-problematic forms of self reference. Additionally, carefully explained examples are used to explain the details of valid and invalid forms of self reference. These arguments lead to the introduction of an axiom schema that does not sacrifice any of ZFC's benefits and allows for self reference and self containment.

First, we closely outline how functions and sets behave and depend on each other. Sets are collections of objects. A useful way to describe a set is by using a binary function φ. Objects that return *true* when they are inputted into φ are in the set. This is the essence of the axiom schema of separation. Next, the domain of a function is the set of all objects that can be validly inputted into the function. Thus, a set is defined by a function but a function relies on a set for its definition. Although this appears circular, it is not the case. One can define sets without using functions by simply listing the elements in the set. We can use these sets to make functions and then define more sophisticated sets. Now, we describe how this way of defining sets can be described in the language of the axiom schema of separation.

The set $S = \{1,2\}$ can be alternatively and equivalently defined using the function $\varphi(x)$ that returns true if and only if x equals 1 or 2. Similar functions exist for all sets that are defined by exhaustively listing its elements. For all valid sets, one can use its associated φ function to determine whether any object is in that set. Conversely, every binary function has an associated set that contains all of the objects that return true when inputted into said function.

Many people would object that absolute generality or unrestricted quantification is incorrect. That is, that functions cannot have everything as a domain. I claim that many functions have all objects in their domains. Now, I will give three reasons why some cases of absolute generality are possible. First, claiming that absolute generality is not possible immediately causes a contradiction.

Suppose, unrestricted quantification is not the case. Then, one can not quantify over absolutely everything. The previous sentence is a quantitative statement about everything, which is a contradiction.

Second, many functions such as $O(x) = \varnothing$ are input independent. Thus, anything can be inputted into such a function. Lastly, mathematicians use functions with unrestricted domains without issue.

Other functions do not have everything in their domains. For example, the function $E(x)$ that returns true if x is even only makes sense for integers. Thus, E's domain is the set of integers. However we can define a function φ such that $\varphi(y) = E(y)$ for all y in E's domain. This function is $\varphi(x)$, which returns true if x is a number that has the property of being even. However, some functions cannot be modified to contain a domain with all objects. Objects who rely on a function φ cannot be in the domain of $\varphi(x)$ if the property of x checked by φ relies on φ.

The contradictions that arise from Russell's Paradox stem from incorrectly claiming that R's associated φ function, which returns true when x does not contain itself, has everything in its domain. I claim that R is not in the domain of φ and thus it makes no sense to ask whether R contains itself. Moreover, R is an invalid set because we cannot use its associated φ function on itself. Now we describe a few examples to help describe this issue.

First, we revisit the function $E(x)$. Clearly $E(8) = \varphi(8) =$ *true*, $E(7) = \varphi(7) =$ *false*, and E(Aphrodite) is invalid because Aphrodite is not in the domain of E. Similarly, E(being even) is invalid. Here self reference is obviously nonsensical. Also, note that φ(Aphrodite) = φ(is even) = *false*.

Next, consider the function $L(w)$, whose domain is words and returns true when w has 15 letters. The associated φ function for the set of words that return true when inputted to L is $\varphi(x)$, which returns true if x is a word and has 15 letters. We see that φ(bat) = L(bat) = *false*, φ(objectivenesses) = L(objectivenesses) = *true*, and $L(324)$ is invalid because 324 is not in the domain of L. However, $\varphi(324) =$ *false*. Now, we closely look at φ(fifthteen-lettered) and L(fifteen-lettered).

All words have a property that describes the number of letters that word has. For example, bat has 3, objectivenesses has 15, and fifthteen-lettered also has 15. Thus, φ(fifthteen-lettered) = *L*(fifthteen-lettered) = *true*. In words, 'fifthteen-lettered' is a fifthteen-lettered word. Here, self reference is valid because the number of letters in 'fifthteen-lettered' is independent of the definition or other properties of 'fifteen-lettered.'

Now, we look at *H*(*x*) whose domain is all adjectives and returns true if *x* is heterological. We define φ as the function that returns *true* for all objects that map to *true* when inputted into *H*. We see that φ(blue) = *H*(blue) = *true*, φ(fifthteen-lettered) = *H*(fifteen-lettered) = *false*, *H*(32) is invalid because 32 is not an adjective, but φ(32) = *false*. Grelling's Paradox arises when one is asked to evaluate *H*(heterological). However, *H*(heterological) is invalid because *H*(*x*) checks whether *x* has the property of being non-selfdescribing. However, heterological, by definition, means non-selfdescribing. Thus, the property of 'heterological' that is checked by *H* depends on H and φ. Thus, it is circular. For the same reason, φ(heterological) is invalid. This kind of self reference is incorrect. Therefore, we must adjust the domain of *H* to be all adjectives whose definition is independent of the concept of non-self description. That is, the domain of *H* is all adjectives that are independent of φ. Similarly, the domain of φ is all objects that are independent of φ.

Next, we look at the Liar Paradox. It concerns the statement, "This sentence is false," and asking whether it is true or not. Let *L* be the statement and *T*(*x*) be a function that returns true if *x* is a true statement. Following the results from before, it is clear that the domain of *T* is all statements whose content is independent of φ where φ(*x*) is the associated function of the set of objects that return *true* when inputted to *T*. It is crucial to note that *T* and φ are different functions with different domains.

Now, we evaluate *T*(x) and φ(*x*) for some values. Clearly, *T*(2+2=4) = φ(2+2=4) = *true* and *T*(1+1=5) = φ(1+1 = 5) = *false*. Then, φ(apple) = *false*, but *T*(apple) is invalid as 'apple' is not a complete statement and thus is not in the domain of *T*. However, T(*L*) and φ(*L*) are both invalid as *L* is not in the domain of *T* nor φ because *L* is a statement that depends on φ. Further analysis on the Liar Paradox is given further below when discussing the validity of definitions.

Finally, we examine Russell's Paradox. Let *F* be the same as the second paragraph and define φ as before. We can see that φ({8,9}) = *F*({8,9}) = *true*, φ({x : x=x}) = *F*({x : x=x}) = *false*, *F*(cats) is invalid since 'cats' is not a set and thus not in the domain of F. However, φ(cats) = *false*. Evaluating *F*(*R*) is Russell's Paradox.. *F*(*x*) checks the property of non self containment and so does φ. By definition, *R* is the set of all objects that have the property of non self containment. Thus, the property of *R* that is checked by *F* relies on φ. Thus, *F*(*R*) and φ(*R*) are invalid.

One could argue that the definition of *x* is not the property checked by *F*(*x*). Instead, *F* checks whether any element of *x* is equal to *x*. Then, *F*(*R*) does not depend on the definition of *R* but on its elements. Although this argument holds, checking whether an element of *x* is equal to *x* depends on all the properties of *x* because to check for equality between two objects, one must look at every property of the objects. Clearly, the definition of *x* is a property of *x*. Thus, *F*(*R*) is invalid. It follows that *R* is an invalid set because all valid sets have φ functions that can take any object as input and φ(*R*) is invalid.

These four examples illustrate how self reference is valid when the property of an object that is checked by a function does not rely on the very function. Sets that are self referential are forbidden in ZFC. However, we just proved that some of them are valid. For example the set of all sets, *S*, includes itself. The property checked by its φ function is the property of being a set. The property of *S* being a set does not rely on φ in the same way that the property of *R* not containing itself relies on its associated φ function. The difference is that *R*'s property and φ relation is circular while S's is not. We can verify that S has the property of being a set directly from its definition. That is, S asserts that it has the property of being a set. Thus, we know that $\varphi(S) = true$. On the other hand, R's definition points to its φ function without clearly stating what $\varphi(R)$ evaluates to. Thus, $\varphi(R)$ is circular and invalid. This shows that self reference is only valid when an object's property checked by φ is independent of φ or if the object's definition validly states the value of φ(object). By 'validly states,' I mean that the statement does not cause a contradiction.

An example of a definition that invalidly states its φ value is the Liar Paradox. Clearly it states that $\varphi(L) = false$. However, the definition of L is invalid because statements implicitly claim that they are true. Thus L's definition claims $\varphi(L) = true$ and $\varphi(L) = false$. This is a contradiction, thus L is an invalid statement. Hence, self reference and containment are possible in consistent systems.

Since self reference and self containment are valid, we should modify contemporary ZFC axioms to allow for such sets. I propose adding a new axiom schema. In mathematical notation, $\forall\ \varphi{:}D \rightarrow \{0,1\}\ \ \exists A : A = \{x : \varphi(x) = 1\}, D = \{x : (P(x) \perp\!\!\!\perp \varphi) \vee (D(x) \ni \varphi(x))\}$ where $P(x)$ is the property of x checked by φ and $D(x)$ is the definition of x. In English words, given a binary function φ, there exists a set A such that the elements of A are all the objects that return true when inputted into φ. The domain of φ is all objects whose property checked by φ is independent of φ and all objects whose definition validly includes its value of $\varphi(x)$. It is important to note that this is not the same as the axiom schema of separation since it does not deal with subsets.

This schema gives mathematicians, logicians, and philosophers the liberty to use valid self containing sets and self reference, which is prohibited by the ZFC axioms. Additionally, this new schema only requires abandoning the axiom of regularity, which only exists to avoid self containment. Thus, no significant sacrifices are made by accepting this schema. Additionally, this schema addresses the problems that arise from Russell's Paradox and other paradoxes of self reference and carefully explains when self reference can be used.

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