Laplace Transforms in Differential Equations

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I. Introduction

The Laplace Transform is an integral transform that, much like the Fourier Transform, is a very powerful tool mathematicians use to solve differential equations. However, it can be more general than the Fourier Transform, as it allows complex numbers to have a non-zero real value and includes all of the initial conditions during the transformation. Consequently, the Laplace Transform is applicable to many fields of science. It would be of great benefit to explore the definitions, properties, and applications of the Laplace transform, as well as its use in solving partial differential equations and a brief history of its purpose and creation.

A History of the Laplace Transform

Since its inception in the late 1700s, the Laplace Transform has been instrumental in solving certain classes of partial differential equations. In recent years, it has become a powerful tool in many applications of differential equations, allowing solutions to systems which would otherwise be very difficult to solve. First invented by Pierre-Simon Laplace, and later developed by Olivier Heaviside near the turn of the 20th century, the Laplace transform is commonly used in applications relating to electrical circuits or similar fields, and is a staple of electrical engineering (Britannica - Laplace transform). The legendary mathematician Johannes Euler had begun seeking solutions of the type provided by the Laplace Transform to certain differential equations as early as 1744 in *De Construction Aequationum,* and his work was finished by the also legendary Laplace decades later. Euler's intended purpose was to find solutions to equations of the form:

$$z = \int_0^x e^{at} x(t) dt$$

(Deakin 346-347)

This eventually led to the transformation used to solve such equations and a myriad of technological advances in the fields of mathematics and electromagnetism. To see how such an equation could be solved, it would be beneficial to examine the definition and practical application of the Laplace Transform.

II. Background and Using the Laplace Transform

Defining and Using the Laplace Transform

Definition 2.1

The Laplace Transform operator applied to a piecewise continuous function *u* is written as (***L***u)(s) or U(s), and is defined to be the following integral over the time domain:

$$(\mathcal{L}u)(s) = \int_0^\infty u(t) e^{-st} dt$$

(Logan 106)

This transform converts a function in the time domain into a function in the transform domain, where u and s are known as transform variables.

Further Explanation of the Transform:

Much how it is true that any wave can be expressed as a sum of sine functions, every wave can be expressed as a sum of exponential functions of the form

$$(c + di)e^{a + bi}$$

Essentially, The Laplace Transform will tell us how much of each function we have to add.

Now, let every function we wish to add be a point in a plane of functions. Then, let the distance between the functions we want to add, $ds \to 0$. Then, we can write the sum as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds = f(t)$$

(Khutoryansky)

Later, we will define this equation as the Inverse Fourier Transform. Solving for the Laplace Transform requires knowledge of complex variables (Logan). However, what we have done is enough to give one an idea of where the Laplace Transform comes from.

Note: The Fourier Transform is very similar but with sine functions instead of exponentials where the real value of the complex constant is zero. Thus, the Laplace Transform is more general.

Properties of the Laplace Transform

Property 2.1.

The laplace transform is linear, as integration is linear on integrable functions. As such, the following equation of linearity holds:

$$\mathcal{L}(cu_1 + u_2) = c\mathcal{L}u_1 + \mathcal{L}u_2$$

(Logan 106)

Property 2.2.

Partial differential equations in time and space have a useful property: their spatial derivatives can be taken outside of the Laplace Transform. This allows us to take the spatial derivative separately and solve the result as an ordinary differential equation.

$$\mathfrak{L}\{w_x(x,t)\} = \frac{\partial}{\partial x}\mathfrak{L}\{w(x,t)\}$$

Property 2.3.

The aspect of the Laplace Transform that makes it a very powerful tool is its ability to convert derivatives of a function in the time domain to simple multiplication operations in the transform domain. Two examples from *Applied Partial Differential Equations Third Edition* by J. David Logan on page 107 are given by:

$$(\mathcal{L}u')(s) = sU(s) - u(0)$$

$$(\mathcal{L}u'')(s) = s^2U(s) - su(0) - u'(0)$$

It may be useful to note that there also exists a generalized form of this result, which is easily shown by mathematical induction and integration by parts:

$$\mathcal{L}\left(u^{(n)}(t)\right) = s^n U(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \dots - u^{(n-1)}(0)$$

(Dyke 15)

This can be written more concisely as:

$$\mathcal{L}\left(u^{(n)}(t)\right) = s^n U(s) - \sum_{i=1}^{n} s^{n-i}u^{(i-1)}(0)$$

This provides the ability to convert a partial differential equation into an ordinary differential equation by eliminating its derivatives. The resulting differential equation can then be solved by conventional means, but will still exist in the transform domain. As such, a conversion formula to return the equation back to the time domain is necessary. This operator is given by the following formula:

Definition 2.2

The Inverse Laplace Transform of a function U(s) is

$$u(t) = \left(\mathcal{L}^{-1}U\right)(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} U(s)\mathrm{e}^{st}ds$$

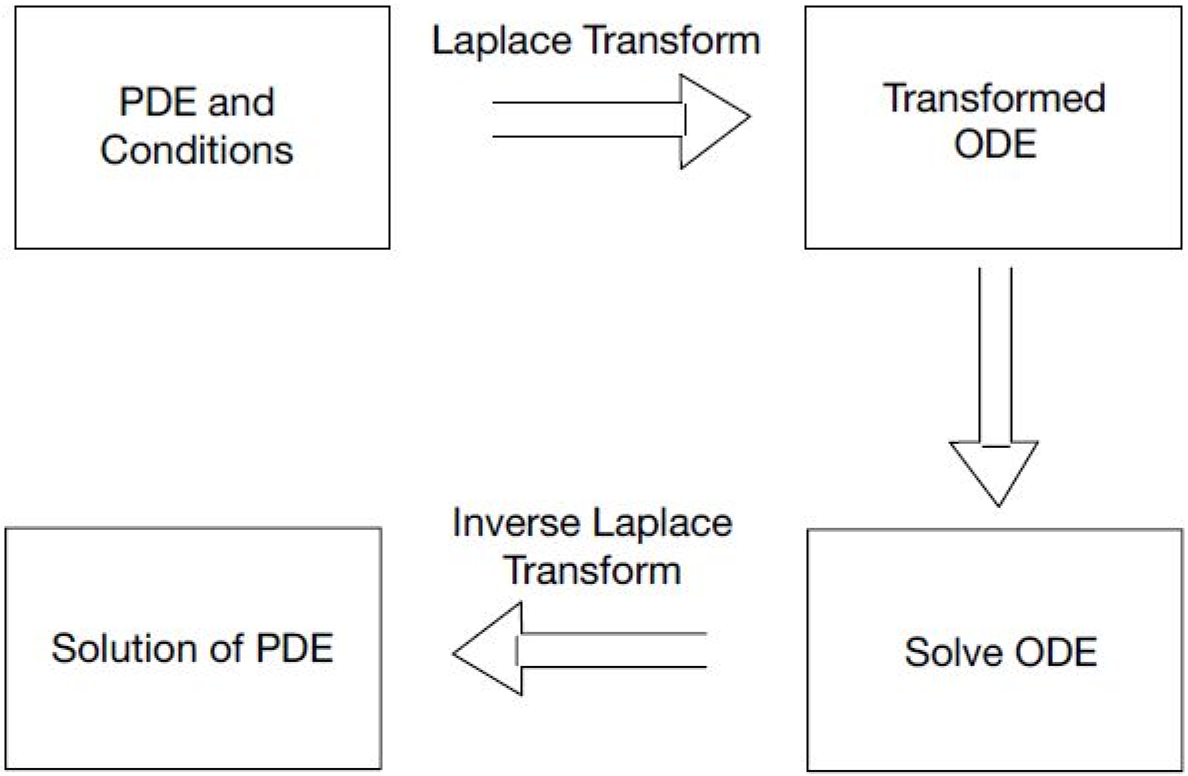
(Logan 107)

The inversion of the Laplace Transform can be done after solving the transformed equation, and will give the desired expression in the time domain. To illustrate the power of the Laplace Transform, we shall observe the following example of its use on an ordinary differential equation:

$$u''' + u'' = 0,\ t > 0$$
$$u''(0) = 0,\ u'(0) = 0,\ u(0) = 1$$

$$\mathcal{L}(u''' + u'') = 0$$
$$s^3U(s) - s^2u(0) - su'(0) - u''(0) + s^2U(s) - su(0) - u'(0) = 0$$
$$s^3U(s) + s^2U(s) - s^2 - s = 0$$
$$s^2(s+1)U(s) = s^2 + s$$
$$U(s) = \frac{s(s+1)}{s^2(s+1)} = \frac{1}{s}$$
$$\text{By Computed Table Values:}$$
$$u(t) = \mathcal{L}^{-1}\big(U(s)\big) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

There are many tables of common inverse Laplace Transforms, and we have used the table from *Applied Partial Differential Equations* on page 114 to solve this example. The solution to this ordinary differential equation is trivial, but provides an example of how Laplace Transforms are used in solving differential equations. The Laplace transform can also be very useful in solving linear partial differential equations with constant coefficients where the coefficients may be complex. Intuition on how the Laplace Transform is used to solve partial differential equations is made clear by the following diagram:



Note that this is the same procedure as solving partial differential equations by using the Fourier transform. This process is very important in a variety of fields, but perhaps equally important is a theorem which forms another useful tool related to the Laplace Transform.

Theorem 2.1. (The Convolution Theorem)

The Convolution Theorem provides a method by which two functions multiplied together in the transform domain can be converted to the convolution of their inverse Laplace Transforms, or vice versa. This is particularly useful when converting a solved system in the transform domain back into the time domain when that system consists of two multiplied functions of the transform variable "s". This property is exemplified by the following system for solving differential equations with the Convolution Theorem:

$$\mathcal{L}\big(u * v\big)(s) = U(s)V(s)$$

$$\mathcal{L}^{-1}\big(U(s)V(s)\big) = \big(u * v\big)(t)$$

$$\big(u * v\big)(t) = \int_0^t u(t-\tau)v(\tau)d\tau$$

(Logan 108)

Since obtaining the product of two functions as the result of a Laplace Transform is not uncommon, it is useful to note that such a result can then be readily converted to a convolution in the time domain. To provide greater insight into the use of this theorem, we have constructed an example of its use:

$$u' + u'' = 0,\ u(0) = 0,\ u'(0) = 1$$

$$\mathcal{L}(u' + u'') = 0$$

$$sU(s) - u(0) + s^2U(s) - su(0) - u'(0) = 0$$

$$s(s+1)U(s) = 1$$

$$U(s) = \frac{1}{s(s+1)} = \frac{1}{s} \cdot \frac{1}{s+1}$$

$$\text{Let } U_1 = \tfrac{1}{s} \text{ and } U_2 = \tfrac{1}{s+1}$$

$$\mathcal{L}^{-1}\big(\tfrac{1}{s}\big) = 1 \text{ and } \mathcal{L}^{-1}\big(\tfrac{1}{s+1}\big) = \mathrm{e}^{-t}$$

By Convolution Theorem:

$$\mathcal{L}\big(U_1 \cdot U_2\big) = \big(u_1 * u_2\big)(t)$$

$$u(t) = \int_0^t u_1(t-\tau)u_2(\tau)d\tau$$

$$u(t) = \int_0^t 1 \cdot \mathrm{e}^{-\tau} d\tau$$

$$u(t) = -\mathrm{e}^{-t}\Big|_0^t$$

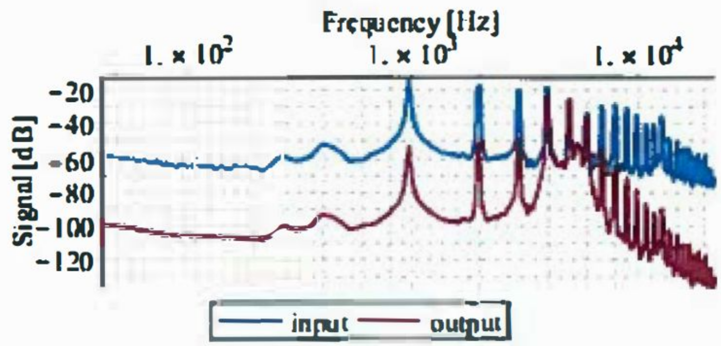
$$u(t) = -\mathrm{e}^{-t} + 1$$

III. Applications

Common Applications of the Laplace Transform

The Laplace Transform has a wide range of applications. We will be focusing on the most common one, which is electrical engineering. Additionally, we will explore some unexpected applications, such as image recognition and neural networks.

The Laplace Transform is critical for electrical engineering in the subfield of signal processing. This is because the Laplace Transform takes in a time domain function and outputs a frequency domain function. For example, we use devices like our cell phones everyday that take in sound waves, turn them into electrical signals, and convert them back into the original sound. This process involves changing from time domain functions to frequency domain functions, which is exactly what the Laplace Transform does. The following image shows a sound frequency input and a signal output that was calculated using the Laplace Transform.



Input Wave vs. Output Wave

(Cocagne 8)

Thus, the Laplace Transform can be used in any kind of signal processing. However, this is not its only application.

Another application of the Laplace Transform that is not commonly known is in image recognition. The Laplace Transform can be used to detect the edges of objects in images. In some cases, it is more effective than other image detection methods because it does not require cutting the edges before applying the transform. For example, Anna Gorbenko and Vladmir Popov used the Laplace Transform to create an algorithm that would detect train tracks.

(Gorbenko 2416)

Another unexpected application can be found in economics. It can be used to calculate the present discounted value of an asset based on cash flow that increases by a continuous compounding interest. This is very useful for investors who want to buy assets when there is the highest discount possible (Ananda 360).

IV. Conclusion

The Laplace Transform is instrumental in many areas of mathematics, and readily provides solutions to otherwise complex differential equations, both partial and ordinary. This process is facilitated by the definitions and properties shown in section II. The history of the Laplace Transform spans over 200 years, and continues to grow today, as more uses of this mathematical tool are discovered in fields ranging from electrical engineering to economics. As such, the importance of understanding the Laplace Transform and its functionality relating to differential equations is perhaps greater than ever before.

Works Cited

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