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Game Formalism and Deductivism's Unexpected Commitments

Game formalism and deductivism are philosophies of mathematics that attempt to avoid any meaning beyond what matters for mathematics and mathematicians. This would prevent metaphysical and epistemological problems that other philosophies have, which seems promising because these problems do not seem to hinder our mathematical knowledge. However, game formalism and deductivism fail to achieve this goal because the formalists and deductivists must believe that there are abstract objects that are objective and mind-independent. If not, their theories would result as inconsistent. Although their motivation fails, branches of mathematics appear to be games that humans come up with, much how these two philosophies describe them to be. Thus, it is worthwhile to explore these commitments that the game formalists and deductivists are making.

This essay is structured as follows: first, we will explain the relevant ideas of game formalism and deductivism. Then, we will explain some differences between the views. Second, we will state and use Gödel's Second Incompleteness Theorem to show that these two philosophies run into massive problems. Next, we will show that to avoid these problems, game formalists and deductivists must accept the existence of abstract objects that determine the consistency of mathematical games. Then, we will use this acceptance to explain some common objections to game formalism and deductivism and show that we cannot know every mathematical truth. Finally, we will defend some natural objections one might have against the acceptance of abstract objects.

First, game formalism proposes that all of mathematics is composed of different games and rules that we come up with and then play. In other words, mathematical objects have no meaning beyond their symbol that can only be used according to specific rules. It is important to note that the term 'symbol' is inaccurate because normally, symbols stand for some object. However, here, they do not represent anything.[1] For the formalists, numbers are like chess pieces while equations and theorems are like the moves of the game. In addition, writing a correct equation or using a theorem is like performing a legal move, while correctly calculating something or proving a claim is like winning the game. Then, since mathematics is simply a game, mathematical objects and statements are meaningless or their meaning does not matter for mathematics.[2] Similarly, the meaning of a chess game does not affect chess nor the chess player.

Next, we will notice that some mathematical games are rich and worthwhile while others are boring games where anything can be proven. Generally, interesting mathematical games are consistent. That is, their rules are not contradictory and are true to logic. Non-consistent games can use contradictions to prove any statement. For example, a game where "0 = 0" and "0 ≠ 0" are true will prove that everything is both identical and not identical to

[1] Stewart Shapiro. *Thinking about mathematics: The philosophy of mathematics*. OUP Oxford (2000): 144-167

[2] Alan Weir. "Formalism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Spring 2020 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/spr2020/entries/formalism-mathematics/

itself. Therefore, game formalists rely on the consistency of certain games for the richness of mathematics.

Additionally, game formalists defend the applicability of mathematics in the real world by arguing that the rules of certain games can be applied to the real world. For example, the game of arithmetic is useful because its rules apply to counting and grouping collections of objects. Thus, all of the theorems and equations of arithmetic will apply to counting and grouping objects in the real world. Essentially, to make useful games, we must choose rules that are true in nature.

Second, deductivism is a more sophisticated version of game formalism. This theory was developed by David Hilbert. Much like the formalists, the deductivists believe that mathematics has no meaning beyond its subject. Each branch of mathematics, or game, has arbitrary predetermined axioms and follows the rules of logic. The applicability argument is also similar to the formalist view. If the axioms are true for a system, then all of the theorems of that game are also true for the system.

However, Hilbert disagrees with the formalists on the meaning of mathematical statements. He claims that they are a group of symbols that logically follow from the axioms, which are another set of symbols, instead of moves of a game. In deductivism, logic serves as the tool that lets us know the truth value of an otherwise meaningless mathematical sentence. Nevertheless, deductivists, like the formalists, rely on logic for the richness of mathematics.

To build on and push the ideas of deductivism, Hilbert developed what is known as Hilbert's Program. Zach writes, "[Hilbert's Program] calls for a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent."[3] In other words, Hilbert's Program is an attempt to describe all of the axioms for each branch of mathematics and prove that they are consistent. However, after years of development, Hilbert's Program was abandoned because Gödel's incompleteness theorems showed that such a proof was not possible. Regardless, Hilbert's Program achieved the axiomatization of many branches of mathematics and pushed others to continue his efforts. In fact, contemporary deductivists have developed revised Hilbert programs and believe that the foundations of deductivism are correct.

However, there is a major flaw that stops these two theories from achieving their goals. Logic is used to prove propositions in most, if not every, branch of mathematics. Thus, logic must be part of the rules of mathematical games. Also, as explained above, game formalists and deductivists rely on logic to know whether the rules of a game are consistent. Therefore, logic is part of the rules of these games. Thus, they use the rules of the game to prove, or at least believe, that its rules are consistent. However, this is not possible because it violates Gödel's Second Incompleteness Theorem. It states, "No sufficiently strong consistent mathematical theory can prove its own consistency."[4] [5] Thus, to prove that

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[3] Richard Zach, "Hilbert's Program," ed. Edward N Zalta, Stanford Encyclopedia of Philosophy (Stanford University, May 24, 2019), https://plato.stanford.edu/archives/fall2019/entries/hilbert-program.

[4] Thomas Jech. "On Gödel's second incompleteness theorem." *Proceedings of the American Mathematical Society* (1994): 311-313.

[5] Kurt Gödel. *On formally undecidable propositions of Principia Mathematica and related systems*. Courier Corporation, 1992.

mathematics is consistent, the formalists and deductivists must use machinery that is not connected to any part of the game.

There is also a similar problem regarding how mathematicians explain games using theories that are not part of the games themselves. Weir writes, “The problem this raises for the formalist is this: the metatheory is itself a substantial piece of mathematics.”[6] [7] In other words, this out of game knowledge is used by mathematicians to make statements about the game. This is problematic because we should be able to play mathematical games by solely using their rules and pieces. Essentially, these problems are about how mathematics, by itself, cannot show its consistency even though we know that it holds truth. We know this because mathematics is useful.

To avoid these problems, there are two options. First, one could claim that logic, metatheory, and properties about consistent games are rules of an even broader system where mathematical games are pieces of that system. However, this leads to the same problem regarding Gödel’s Second Incompleteness Theorem, since we would need something beyond logic, metatheory, and these properties to prove that the bigger game is consistent. Thus, we move on to our second option, which does not violate the incompleteness theorem.

To make their philosophies consistent, formalists and deductivists must accept that truths about games, their consistencies, and their properties are objective and mind-independent. This is because, any mathematical statement, such as “2 + 2 = 4,” will always be true regardless of who does mathematics. Furthermore, we can only interpret these truths

[6] Weir, “Formalism” 2020

[7] Gottlob Frege. *Grundgesetze der arithmetik*. Vol. 2. H. Pohle, 1893.

through our games and their rules. Thus, these truths must be a result of abstract objects. The acceptance of these abstract objects does not fit the motivation of these philosophies. However, denying or ignoring their existence makes them fall into the problems described above. Therefore, the game formalists and deductivists are committed to accept the existence of some platonic object or objects that allow us to determine the truths and consistency of a game.

Next, the acceptance of these abstract objects helps us further explain areas of game formalism and deductivism that are frequently objected. For example, many mathematical games, whether by intention or coincidence, can be applied to solve problems about the real world, which are not part of the game. Game formalists and deductivists argue that this happens because the rules, or axioms, apply to the real world. However, this leads to epistemological problems such as the question, “Are the rules and axioms of our world predetermined?” The acceptance of abstract objects makes this easy to answer because the abstract objects determine the truths and their applications. Therefore, since abstract objects exist, we do not come up with the rules of our world because they are predetermined. This is consistent with the way we perceive the world because, in science, we try to find these rules. Similarly, the abstract objects also predetermine which games are mathematically rich and which ones are boring.

Since the abstract objects are out of our reach, it is impossible to properly use them to prove that a game is consistent. Moreover, these objects describe games that we cannot play. If abstract objects describe every mathematical game, then they describe a game where we are the pieces. However, we would not be able to control this game because we are part of it.

For example, a football player cannot control everything that happens in a game of football because they are simply a part of the game. However, the mathematician has complete control over the branch of mathematics that they study. In other words, the concept of playing games like football is different to playing math. So, any consistent mathematical game where humans are the pieces cannot be studied by humans the way we study mathematics. Thus, every truth about every game is not accessible to us. Although this is not the statement of the incompleteness theorems, it agrees with them and possibly serves as an additional reason of why they and their proofs are true.

Since we can only access these objects via our mathematical games, whenever we focus on a certain game, the objects allow us to study the game's metatheory. This makes sense because any abstract object can only be studied indirectly through what we are physically or mentally focusing on. Alternatively, the abstract objects will only let us know about the game we are focused on.

Now, we will consider some common objections one might have to accepting the existence of abstract objects in game formalism and deductivism. The most fundamental argument one can make is to claim that this acceptance is against the goals of the philosophies so one must either deny this claim or move on to a new philosophy. That is, accepting the existence of abstract objects that give meaning to mathematical games is against the spirit of game formalism and deductivism, which is to avoid giving any meaning to mathematics and that this should not be considered.

However, a different interpretation of the goal allows for such abstract objects to be part of the philosophies. It is true that any potential meaning of mathematics does not affect the mathematician's work. While this may be true, this does not mean that avoiding any meaning will produce a correct philosophy of mathematics. Instead, we should build a philosophy of mathematics where any meaning beyond the subject does not affect the practice of mathematics. Both, game formalism and deductivism fit this description.

Additionally, accepting the existence of abstract objects also achieve this because any debate over what these objects are, their behavior, or anything about them will not affect the mathematician's methods. This is because the only thing about the objects that affects the mathematician is the fact that because of them, the mathematician can determine whether a game is consistent or not. The mathematician need not know what is going on under the hood if they accept that the abstract objects allow them to know that a game is consistent. This preserves the spirit of formalism and deductivism in the sense that any meaning beyond mathematics should not and does not matter to the mathematician and their subject.

Another objection one might have to the acceptance of abstract objects is that there is seemingly no reason to take the objects to be mind-independent. However, since physical objects are independent of humans, then objective truths about them are also independent of humans. Then, if we make a mathematical game where the axioms apply to the physical object, the abstract objects' preexisting truths would apply to the physical object. Thus, since the object is independent of us and the abstract objects tell us preexisting objective truths about it, then the abstract objects must also be independent of us.[8] For example, the planet Saturn would exist even if no intelligent life could notice it. Evidently, any mathematical

[8] Øystein Linnebo. "Platonism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/spr2018/entries/platonism-mathematics/

statements about Saturn like its number of moons, number of rings, or shape would be true regardless of intelligent life. Thus, if the objects exist, which we have argued why this should be accepted, then they must also be mind-independent.

Game formalism and deductivism fail because of Gödel's Second Incompleteness Theorem and metatheory problems. However, if these philosophies commit to the existence of abstract objects that determine objective truths and consistency of every mathematical game, then there would no longer be an issue of using mathematics to prove its own consistency. Additionally, this acceptance is in line with the primary motivation of these views. Namely, that the results and practice of mathematics are not affected by any meaning beyond its subject.