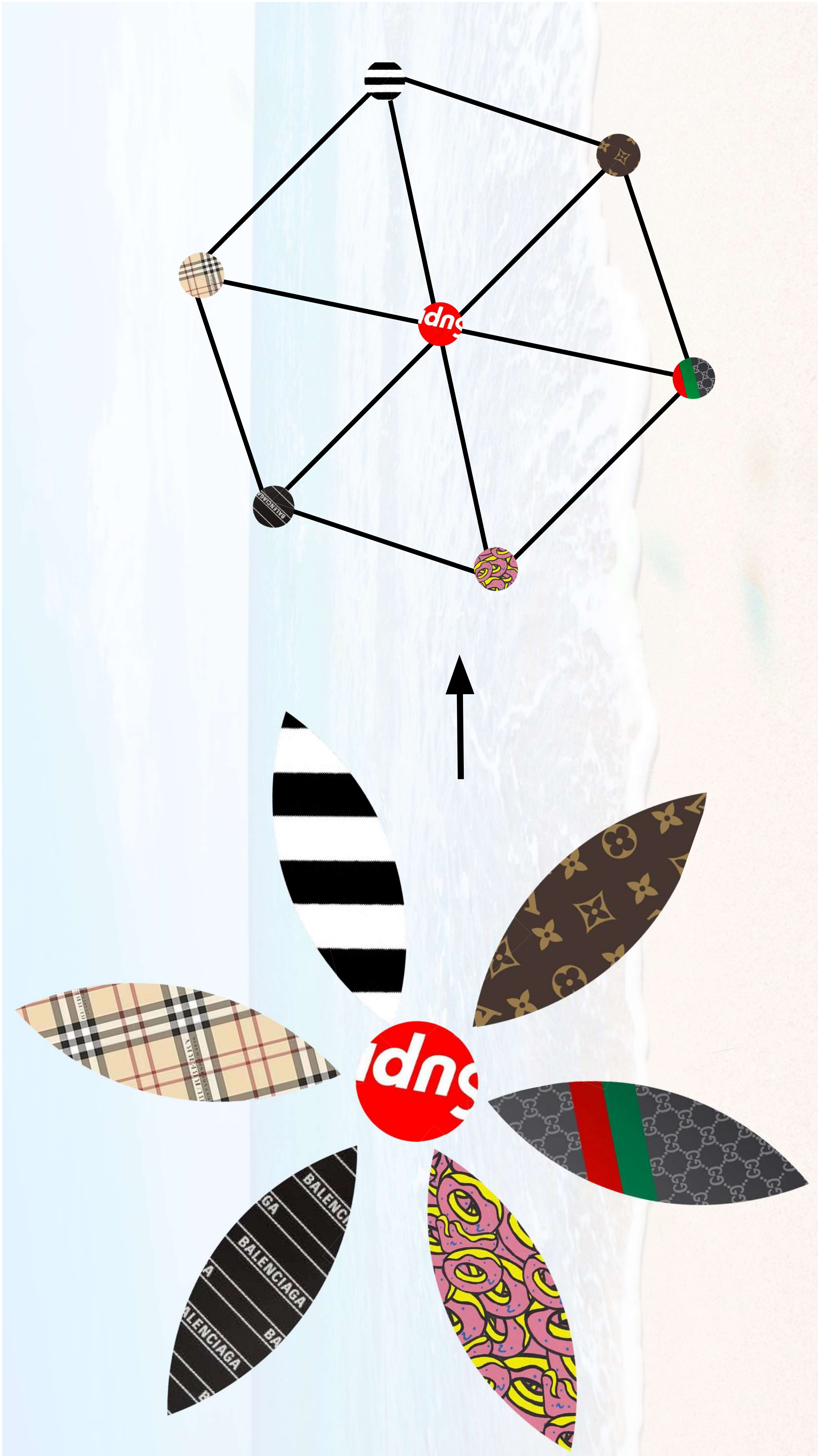
Artisan Beach Balls

Sameerah Munshi & Arturo Ortiz

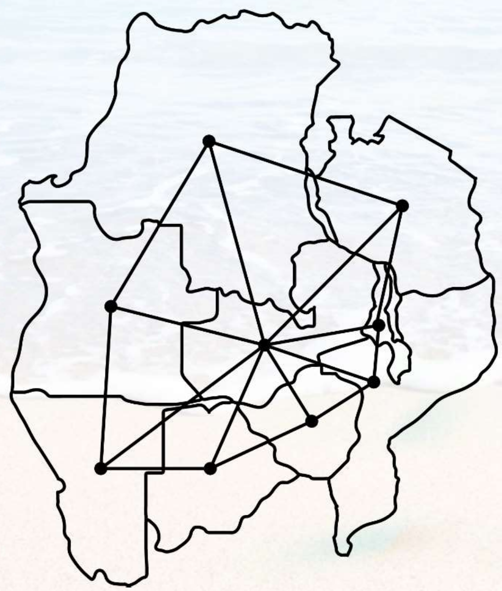
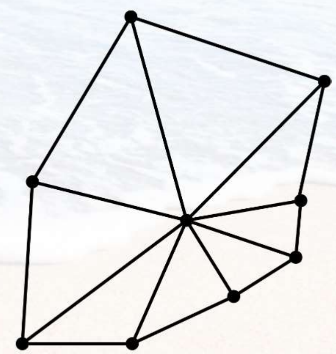
Graph Theory Problem

What is the smallest coloring for a simple planar graph?



Why planar?

- Intersecting edges cannot exist because a border cannot intersect another border.

Four Color Theorem

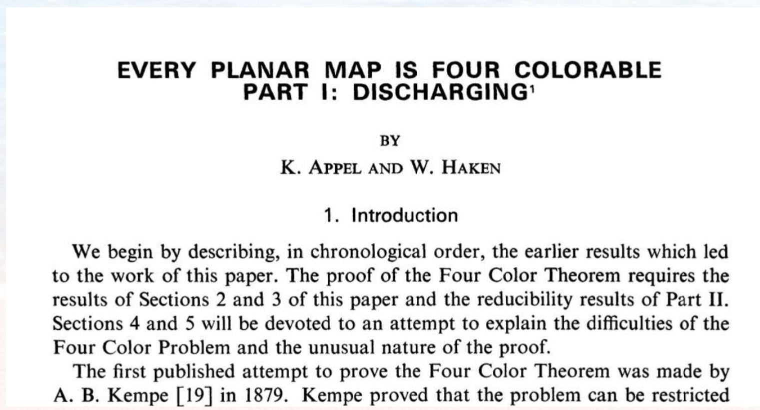
Any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color.

Guthrie's Problem

- Francis Guthrie in 1852
- Augustus de Morgan

A Proof

- Appel, Haken, and Koch in 1977
- First ever computer-assisted proof
- *Controversial*



EVERY PLANAR MAP IS FOUR COLORABLE PART I: DISCHARGING[1]

BY

K. APPEL AND W. HAKEN

1. Introduction

We begin by describing, in chronological order, the earlier results which led to the work of this paper. The proof of the Four Color Theorem requires the results of Sections 2 and 3 of this paper and the reducibility results of Part II. Sections 4 and 5 will be devoted to an attempt to explain the difficulties of the Four Color Problem and the unusual nature of the proof.

The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted

A Proof

- If false, there must exist at least one map that requires 5 colors
- Proof shows that this counterexample cannot exist
- Unavoidable Set
    - set of configurations (subgraphs) from which any planar graph has at least one member of the set as a subgraph.
- Reducible Configuration
    - subgraph if contained in a graph, any coloring of the rest of the graph can be extended into a coloring of the entire graph.

A Proof

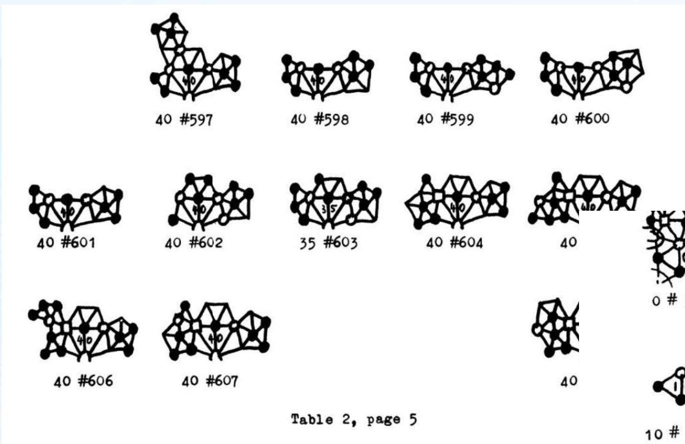

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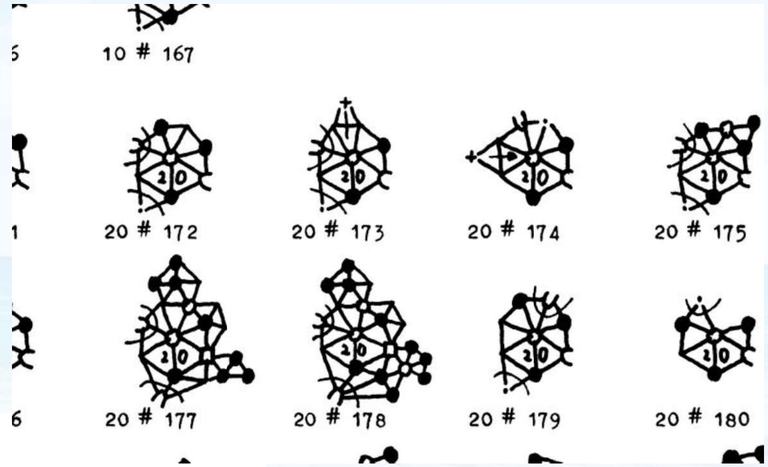
Table 2, page 5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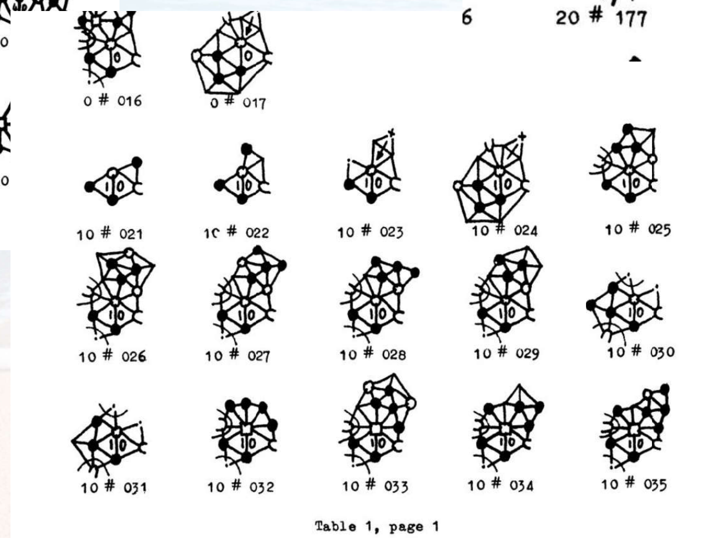
20 # 174
20 # 175
20 # 177
20 # 178
20 # 179
20 # 180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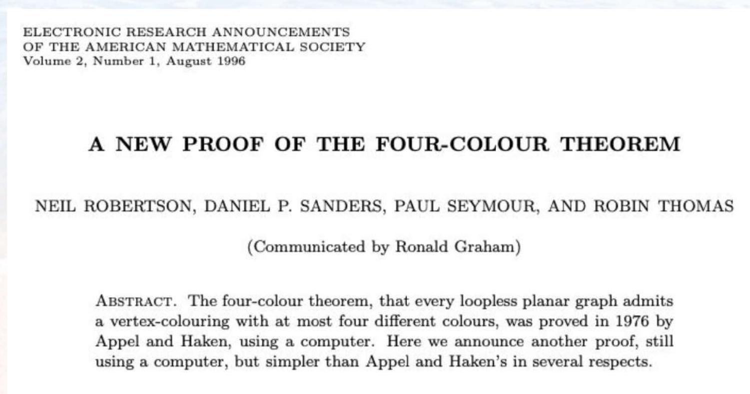
0 # 017
10 # 021
1C # 022
10 # 023
10 # 024
10 # 025
10 # 026
10 # 027
10 # 028
10 # 029
10 # 030
10 # 031
10 # 032
10 # 033
10 # 034
10 # 035
Table 1, page 1

A Proof…Again

- Robertson, Sanders, Seymour, and Thomas in 1997
- From 1936 to 633
- Runs for 3 hours



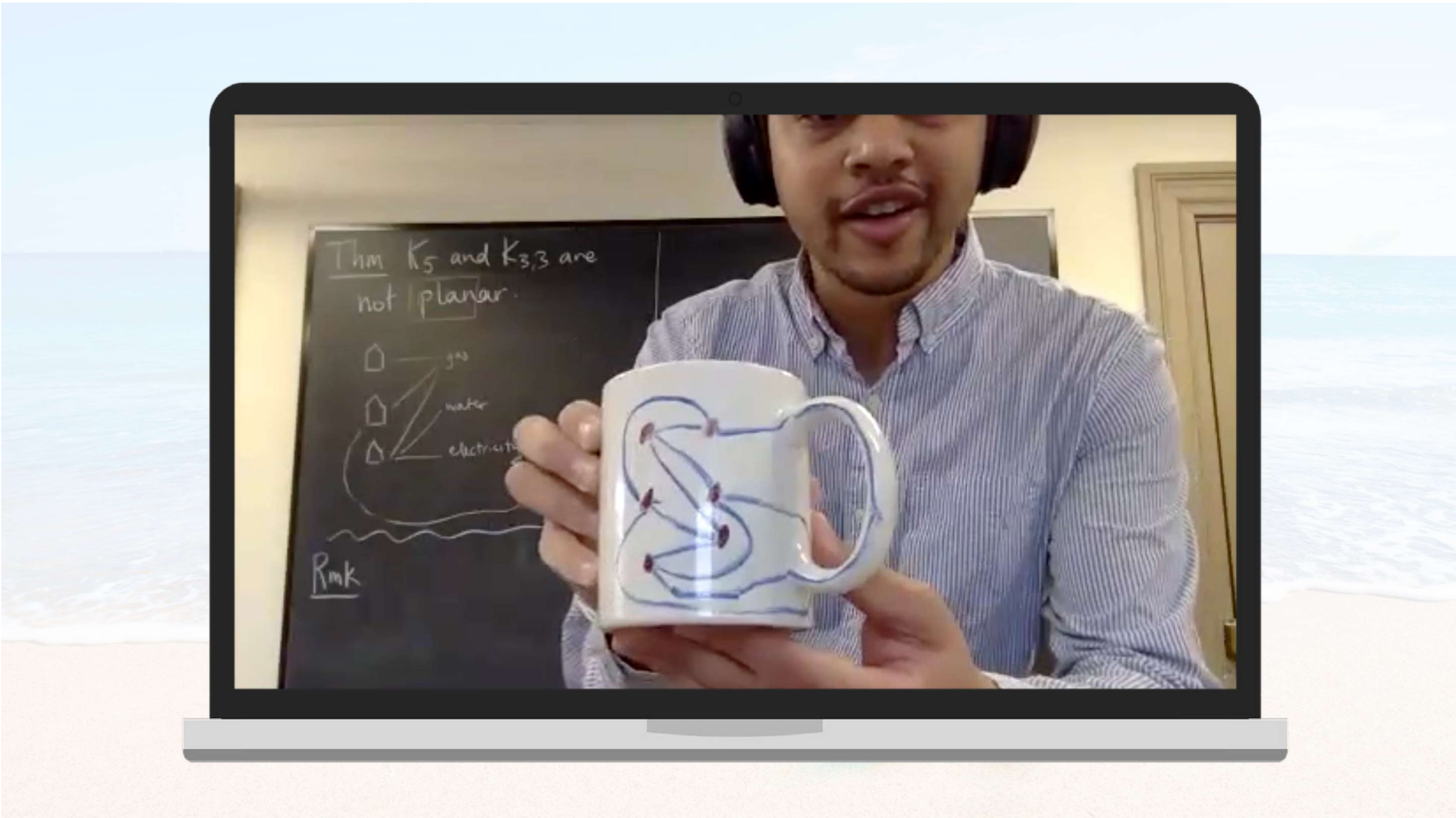
ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, August 1996

A NEW PROOF OF THE FOUR-COLOUR THEOREM

NEIL ROBERTSON, DANIEL P. SANDERS, PAUL SEYMOUR, AND ROBIN THOMAS

(Communicated by Ronald Graham)

Abstract. The four-colour theorem, that every loopless planar graph admits a vertex-colouring with at most four different colours, was proved in 1976 by Appel and Haken, using a computer. Here we announce another proof, still using a computer, but simpler than Appel and Haken's in several respects.


Thm K5 and K3,3 are not planar.
gas
water
electricity
Rmk

Weaker Version

- Alfred Kempe in 1879
- Percy Heawood in 1890

Five Color Theorem!!

Given a partition of regions in a plane, the region can be colored with at most five colors such that adjacent regions do not have the same color.

Alternatively, every planar graph is 5-colorable.

Reminders before proof

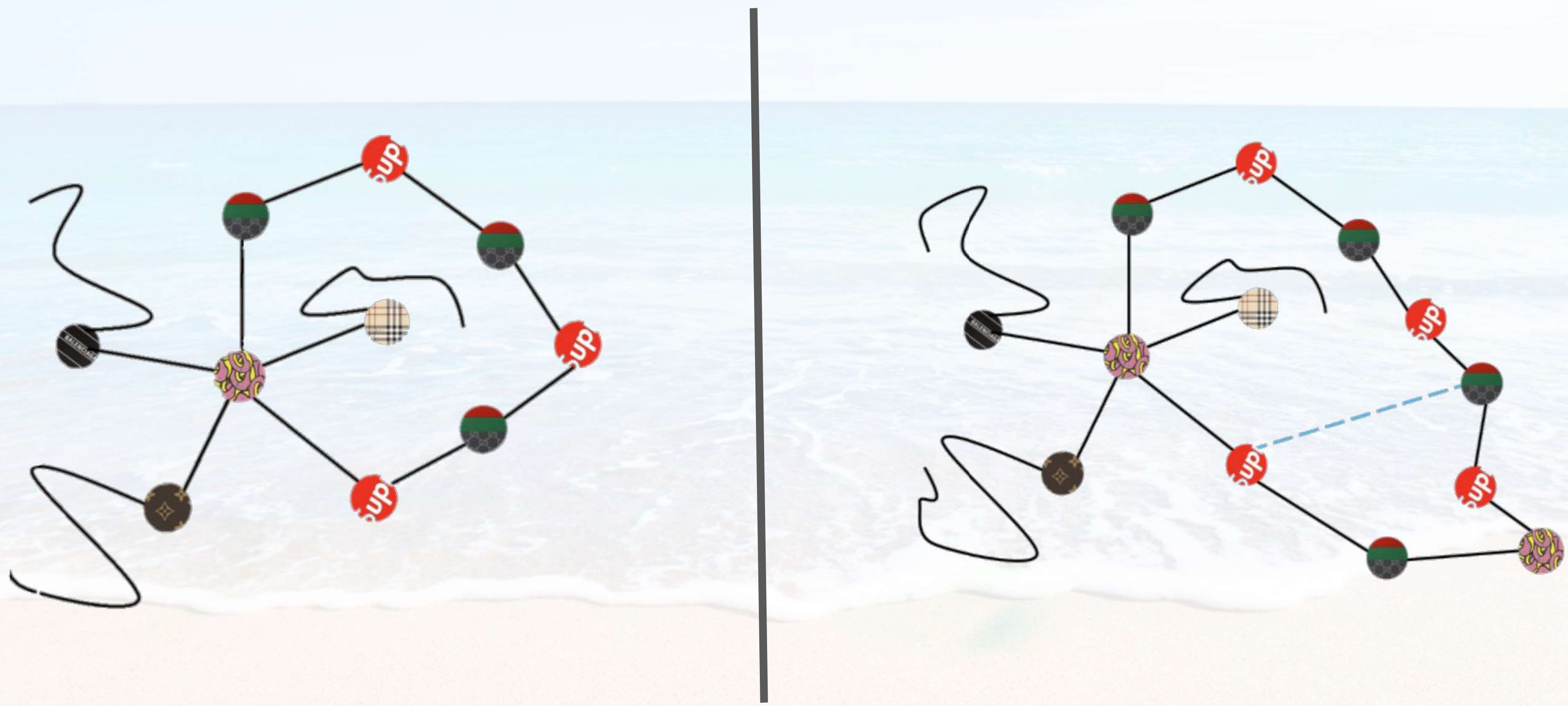
- Six Color Theorem: $|E| = 3n - 6 \rightarrow deg(G) \leq 5$

Proof by Induction

- Base Case: $n = 1$

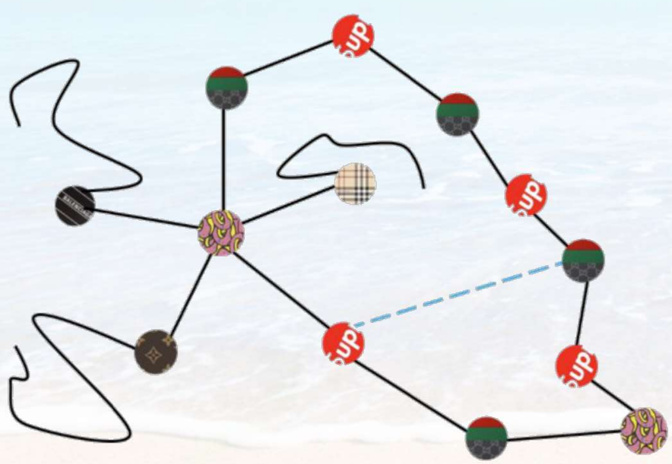
- Induction Step: Assume the Five Color Theorem is true for n vertices

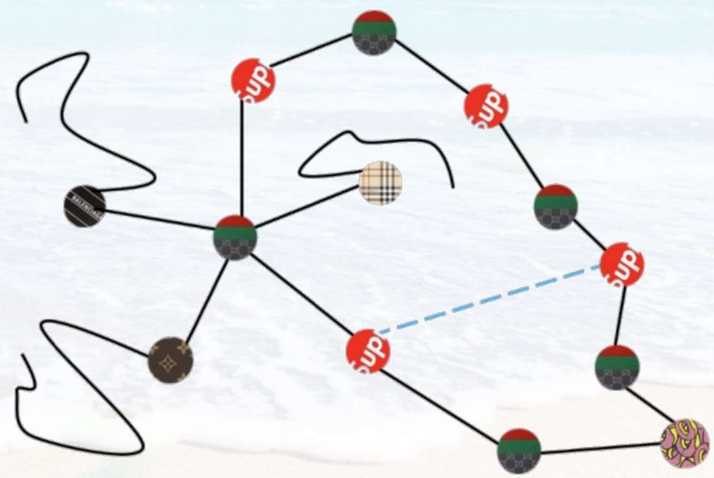
Two Cases



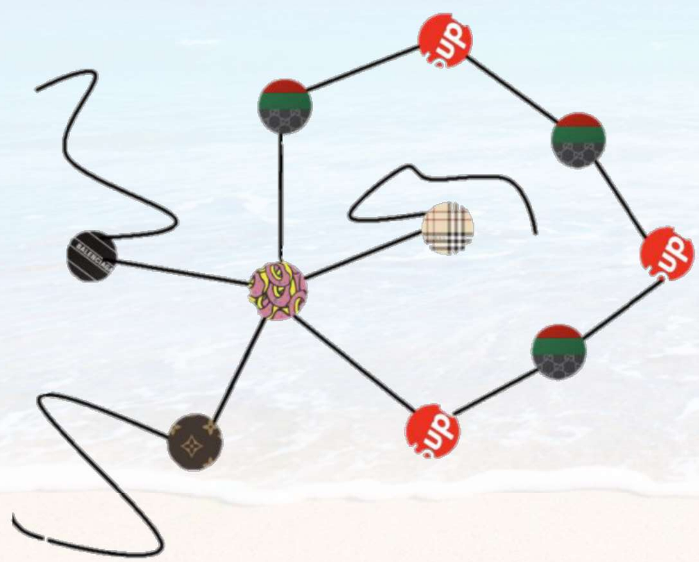
No Alternating Path

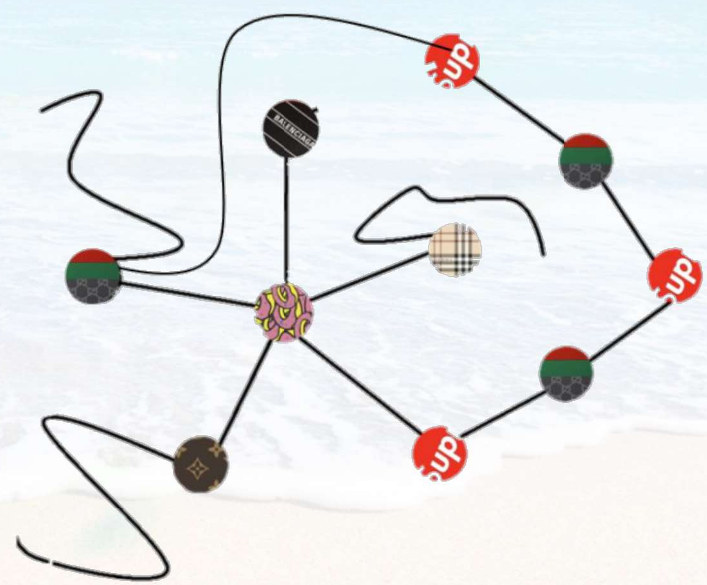
- Just switch!



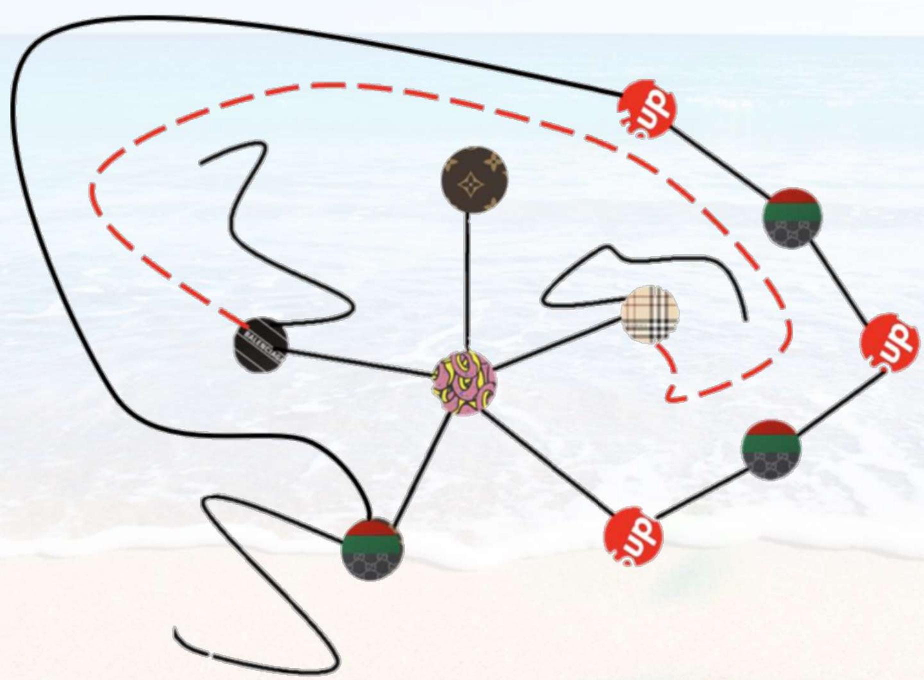


Alternating Path





Alternating Path (cont.)



Algorithm

- There are many algorithms that give a 5-coloring of a planar graph
- The most efficient have O(n) time complexity

$O(n^2)$ Algorithm

- Order vertices by degree. $O(n\log(n))$
- Color v_1 with color 1. $O(1)$
- Let m be the smallest number not connecting to current vertex and let j be current coloring

$O(n^2)$ Algorithm

- If $m \leq j$
  - Color v_i with color j
- Otherwise
  - If for some colors $a,b \subset K$, $a \neq b$, an a,b component of $\langle v_1,\ldots,v_{i-1} \rangle$ has only one vertex adjacent to to v_i in $\langle v_1,\ldots,v_i \rangle$, then perform one a,b interchange on one such a,b component of $\langle v_1,\ldots,v_{i-1} \rangle$. Then, color v_i with the available color, a or b.
  - If no interchange is possible color v_i with $j + 1$

Sources

Performance by Anthony Bosman, Graph Theory 7: Five Color Theorem, Andrews University, 2 Apr. 2020.

Wahab, Matthew. “Algorithm.” Five Color Theorem, McGill University, 2003.

West, Douglas Brent. Introduction to Graph Theory. Pearson, 2018.

“Four-Color Theorem.” Wolfram MathWorld, mathworld.wolfram.com/Four-ColorTheorem.html.


catch u this summer ;)
get vaccinated!